

# Math 525: Assignment 7 Solutions

1. We know  $X_n, Y_n$  converge in distribution to  $X, Y \sim \text{Poisson}(\lambda)$ . Therefore,

$$\phi_{Z_n}(t) = \mathbb{E} [e^{it(X_n - Y_n)}] = \mathbb{E} [e^{itX_n}] \mathbb{E} [e^{-itY_n}] = \phi_{X_n}(t)\phi_{Y_n}(-t) \rightarrow \phi_X(t)\phi_Y(-t)$$

by one direction of Lévy's continuity theorem. Therefore,

$$\phi_X(t)\phi_Y(-t) = \exp(\lambda(e^{it} - 1)) \exp(\lambda(e^{-it} - 1)) = \exp(\lambda(e^{it} - e^{-it})) = \exp(2\lambda(\cosh t - 1)).$$

By the other direction of Lévy's continuity theorem,  $Z_n$  converges in distribution to some random variable  $Z$  with the above characteristic function  $\phi_X\phi_Y$ , as desired.

2. See Exercise 2.2 of Lecture 15.  
 3. To show that  $((X_n, X_{n+1}))_{n \geq 0}$  is a Markov chain, note that

$$\begin{aligned} \mathbb{P}((X_n, X_{n+1}) = (i_n, i_{n+1}) \mid (X_0, X_1) = (i_0, i_1), \dots, (X_{n-1}, X_n) = (i_{n-1}, i_n)) \\ = \mathbb{P}(X_{n+1} = i_{n+1} \mid X_0 = i_0, \dots, X_n = i_n) = \mathbb{P}(X_{n+1} = i_{n+1} \mid X_n = i_n) \\ = \mathbb{P}(X_{n+1} = i_{n+1} \mid X_{n-1} = i_{n-1}, X_n = i_n) \\ = \mathbb{P}((X_n, X_{n+1}) = (i_n, i_{n+1}) \mid (X_{n-1}, X_n) = (i_{n-1}, i_n)). \end{aligned}$$

This follows from

4.  
 (a) If  $P$  is a transition matrix, then
- $$\sum_j (P^2)_{ij} = \sum_j \sum_k P_{ik}P_{kj} = \sum_k \sum_j P_{ik}P_{kj} = \sum_k P_{ik} \sum_j P_{kj} = \sum_k P_{ik} = 1.$$
- (b) If  $P$  is a bistochastic matrix, then  $P$  and  $P^\top$  are transition matrices. Then, by part (a),  $P^2$  and  $(P^\top)^2 = (P^2)^\top$  are transition matrices. Therefore,  $P^2$  is bistochastic.
5.  
 (a) The multiplicity of the eigenvalue is 2.  
 (b) The multiplicity of the eigenvalue is 1.  
 (c) An “end” state of the game is called an *absorbing state*. Since there is a walk from any state  $u$  to an absorbing state  $v$ , the multiplicity of the eigenvalue 1 tells us how many absorbing states there are.