Math 525: Assignment 6 Solutions

1. Below are two ways of solving the problem. The second is more general, in that it works for random variables which do not admit probability densities also.

   (a) Note that

$$\overline{\phi(t)} = \mathbb{E}\left[e^{-itX}\right] = \int_{-\infty}^{\infty} f(x)e^{-itx}dx = \int_{-\infty}^{0} f(x)e^{-itx} + \int_{0}^{\infty} f(x)e^{-itx}dx$$
$$= \int_{-\infty}^{0} f(-x)e^{-itx} + \int_{0}^{\infty} f(-x)e^{-itx}dx.$$

   Make the substitution $y = -x$ to get

$$\int_{-\infty}^{0} f(-x)e^{-itx} = -\int_{\infty}^{0} f(y)e^{ity}dy = \int_{0}^{\infty} f(y)e^{ity}dy$$

   and

$$\int_{0}^{\infty} f(-x)e^{-itx}dx = -\int_{0}^{-\infty} f(y)e^{ity}dy = \int_{-\infty}^{0} f(y)e^{ity}dy.$$

   Therefore,

$$\overline{\phi(t)} = \int_{0}^{\infty} f(y)e^{ity}dy + \int_{-\infty}^{0} f(y)e^{ity}dy = \int_{-\infty}^{\infty} f(y)e^{ity}dy = \mathbb{E}\left[e^{itX}\right] = \phi(t).$$

   (b) If f is even, then

$$\mathbb{P}(X \le x) = \mathbb{P}(X \ge -x)$$

   since

$$\mathbb{P}(X \ge -x) = \int_{-x}^{\infty} f(x)dx = \int_{-x}^{\infty} f(-x)dx = -\int_{x}^{-\infty} f(y)dy = \int_{-\infty}^{x} f(y)dy.$$

   Therefore, X and $-X$ have the same distribution function. In fact, for any random variable X such that X and $-X$ have the same distribution function,

$$\overline{\phi(t)} = \mathbb{E}\left[e^{-itX}\right] = \mathbb{E}\left[e^{itX}\right] = \phi(t).$$

2. Suppose $X_n \to X$ a.s. There are various ways to reach the conclusion. Here are two:

   (a) $\mathbb{E}[e^{itX_n}] \to \mathbb{E}[e^{itX}]$ by the DCT.

(b) Since $X_n \xrightarrow{\mathcal{D}} X$, $\mathbb{E}\left[e^{itX_n}\right] \to \mathbb{E}\left[e^{itX}\right]$ since the function $x \mapsto e^{itX}$ is continuous and bounded.

3. As per the hint,

$$|\phi(t)|^2 = \phi(t)\overline{\phi(t)} = \mathbb{E}\left[e^{itX}\right]\mathbb{E}\left[e^{-itX}\right].$$

Let Y be a random variable with the same distribution as X but which is independent of X. Then,

$$|\phi(t)|^2 = \mathbb{E}\left[e^{itX}\right]\mathbb{E}\left[e^{-itY}\right] = \mathbb{E}\left[e^{it(X-Y)}\right].$$

Therefore, $|\phi|^2$ is the characteristic function of $X - Y$.

4. Suppose X_n and Y_n converge in probability to X and Y, respectively. Note that

$$|X_n + Y_n - (X + Y)| \leq |X_n - X| + |Y_n - Y|.$$

Therefore, for any $\epsilon > 0$,

$$\{|X_n + Y_n - (X + Y)| \geq \epsilon\} \subset \{|X_n - X| + |Y_n - Y| \geq \epsilon\}.$$

Moreover,

$$\{|X_n - X| + |Y_n - Y| \geq \epsilon\} \subset \{|X_n - X| \geq \epsilon/2\} \cup \{|Y_n - Y| \geq \epsilon/2\}.$$

Therefore,

$$\mathbb{P}\{|X_n + Y_n - (X + Y)| \geq \epsilon\} \leq \mathbb{P}\{|X_n - X| \geq \epsilon/2\} + \mathbb{P}\{|Y_n - Y| \geq \epsilon/2\} \to 0.$$

This implies that $X_n + Y_n \to X + Y$ converge in probability, which in turn implies that they converge in distribution.