

Math 525: Assignment 5 Solutions

1. This is just a consequence of Cauchy-Schwarz:

$$\mathbb{E}[XYZ] \leq \sqrt{\mathbb{E}[(XY)^2] \mathbb{E}[Z^2]} = \sqrt{\mathbb{E}[X^2] \mathbb{E}[Y^2] \mathbb{E}[Z^2]}.$$

We have used independence in the last equality.

2. Since $x \mapsto e^{\theta x}$ is convex, this is just a consequence of Jensen's inequality:

$$e^{\theta \mathbb{E}X} \leq \mathbb{E}[e^{\theta X}] = M(\theta).$$

3. Let $\epsilon > 0$ and

$$\Lambda_n^\epsilon = \{|X_n|^p \geq \epsilon\}.$$

By Chebyshev's inequality,

$$\mathbb{P}(\Lambda_n) \leq \frac{1}{\epsilon^p} \mathbb{E}[|X_n|^p] \leq \frac{1}{\epsilon^p} f(n).$$

Therefore, $\sum_n \mathbb{P}(\Lambda_n) < \infty$ and hence by Borel-Cantelli,

$$\mathbb{P}(\limsup_n \Lambda_n^\epsilon) = 0.$$

Now, consider

$$\Lambda = \bigcup_{\substack{\epsilon > 0 \\ \epsilon \in \mathbb{Q}}} \left(\limsup_n \Lambda_n^\epsilon \right).$$

If $\omega \notin \Lambda$, $X_n(\omega) \rightarrow 0$, as desired.