Math 525: Assignment 4 Solutions

1. As per the hint,

$$M(\theta) = \mathbb{E}\left[e^{\theta X}\right] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} e^{\theta u} du = \cdots = e^{\theta^2/2}.$$

   Therefore,

$$M'''(\theta) = e^{\theta^2/2}\theta\left(\theta^2 + 3\right).$$

   Evaulating the above at $\theta = 0$, we obtain the third moment: $\mathbb{E}[X^3] = M'''(0) = 0$.

2.

   (a) We showed in class that if $X = Y$ a.s., then $\mathbb{E}X = \mathbb{E}Y$. Let $Y = 0$, so that $\mathbb{E}X = \mathbb{E}0 = 0$ (the last equality follows since $\mathbb{E}0 = \mathbb{E}[0 \cdot 0] = 0 \cdot \mathbb{E}0$).

   (b) As per the hint, let $A_n = \{X \geq 1/n\}$. Note that

$$\mathbb{E}X = \mathbb{E}\left[XI_{A_n} + XI_{A_n^c}\right] \geq \mathbb{E}\left[XI_{A_n}\right] \geq \mathbb{E}\left[\frac{1}{n} I_{A_n}\right] = \frac{1}{n}\mathbb{P}(A_n).$$

   Since $\mathbb{E}X = 0$, the above implies that $\mathbb{P}(A_n) = 0$. Note also that the sets A_n are increasing: $A_1 \subset A_2 \subset \cdots$ Apply continuity of measure to get $0 = \mathbb{P}(A_n) \to \mathbb{P}(\cup_n A_n)$. Moreover, note that

$$\bigcup_n A_n = \{X > 0\},$$

   as desired.

   (c) $XI_E = 0$ a.s., from which the result follows immediately.

3.

   (a) Note that

$$x = \int_0^x 1dy = \int_0^x 1dy + \int_x^\infty 0dy = \int_0^\infty I_{[0,x)}(y)dy.$$

   Plugging in $x = X$ and taking expectations,

$$\mathbb{E}X = \mathbb{E}\left[\int_0^X I_{[0,X)}(y)dy\right].$$

(b) This is just an application of the Fubini-Tonelli theorem (as a technical note, to apply Fubini-Toenlli, we need X to be integrable).

(c) Note that

$$I_{[0,X)}(y) = \begin{cases} 1 & \text{if } y < X \\ 0 & \text{if } y \geq X. \end{cases}$$

Therefore,

$$\mathbb{E}\left[I_{[0,X)}(y)\right] = \mathbb{P}(X > y).$$

(d) Combining our findings

$$\mathbb{E}X = \int_0^\infty \mathbb{P}(X > y)dy = \int_0^\infty (1 - \mathbb{P}(X \leq y))\, dy = \int_0^\infty (1 - F(y))\, dy.$$