Math 525: Assignment 3 Solutions

1.

(a) As per the hint, $(-\infty, x]^c = (x, \infty) = (x, x+2) \cup (x+1, x+3) \cup \cdots$ is just a countable union of open intervals. Therefore, is generated by $\{(-\infty, x] : x \in \mathbb{R}\} \subset \sigma(\mathcal{G})$. Applying $\sigma(\cdot)$ to both sides of this containment,

$$\mathcal{B}(\mathbb{R}) = \sigma(\{(-\infty, x] : x \in \mathbb{R}\}) \subset \sigma(\sigma(\mathcal{G})) = \sigma(\mathcal{G}).$$

Moreover, note that $\mathcal{G} \subset \mathcal{B}(\mathbb{R})$ since any open interval is a Borel set. Once again applying $\sigma(\cdot)$ to both side of this containment,

$$\sigma(\mathcal{G}) \subset \sigma(\mathcal{B}(\mathbb{R})) = \mathcal{B}(\mathbb{R}).$$

(b) We must establish the three properties of a σ-algebra. (i) $\emptyset \in \mathcal{M}$ since $f^{-1}(\emptyset) = \emptyset \in \mathcal{B}(\mathbb{R})$. (ii) Let $B \in \mathcal{M}$. Then, $f^{-1}(B^c) = (f^{-1}(B))^c \in \mathcal{B}(\mathbb{R})$ and hence $B^c \in \mathcal{M}$. (iii) Let $B_1, B_2, \ldots \in \mathcal{M}$. Then, $f^{-1}(\cup_{n\geq 1} B_n) = \cup_{n \geq 1} f^{-1}(B_n) \in \mathcal{B}(\mathbb{R})$ and hence $\cup_{n\geq 1} B_n \in \mathcal{M}$.

(c) This is actually just a special case of Lindelöf's lemma. If you've seen it, you'll know, if you haven't you just might in a future analysis class.

(d) Applying $\sigma(\cdot)$ to both sides of the containment $\mathcal{G} \subset \mathcal{M}$,

$$\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{G}) \subset \sigma(\mathcal{M}) = \mathcal{M}.$$

2. A discrete random variable X is one for which we can find a countable set $\{x_n\}_n$ satisfying $\sum_{n\geq 1} \mathbb{P}(\{X = x_n\}) = 1$. Moreover, we showed in class that $\mathbb{P}(\{X = x_n\}) = F(x_n) - F(x_n-)$. Combining these two facts gives the desired result.

3.

(a) Note that for $x > 0$,

$$F(x) = \int_0^x \lambda e^{-\lambda x} dx = 1 - e^{-\lambda x}.$$

Therefore, for $0 < y < 1$,

$$F^{-1}(y) = -\frac{\ln(1-y)}{\lambda}.$$

Therefore, we can, without loss of generality, define $X = F^{-1}(Y)$ with F^{-1} as above. Note that we have ignored the events $\{Y = 0\}$ and $\{Y = 1\}$ since these occur with probability zero.

(b) We saw in class that X has distribution function F.

4. Let $Y = \min\{X_1, \dots, X_n\}$. We saw in class that when Y is integrable,

$$\mathbb{E}Y = \sum_{n \geq 1} \mathbb{P}\left(\{Y \geq n\}\right).$$

Therefore,

$$\mathbb{E}Y = \sum_{n \geq 1} \mathbb{P}\left(\{\min\{X_1, \dots, X_m\} \geq n\}\right) = \sum_{n \geq 1} \mathbb{P}\left(\{X_1, \dots, X_m \geq n\}\right) = \sum_{n \geq 1} \mathbb{P}\left(\{X_1 \geq n\}\right)^m$$

as desired (with the sum being on the right-hand side being infinite when Y is not integrable).

5. The inequality $|X|^2 \leq |X| + 1$ shows that X^2 is integrable whenever X is (we saw this in class).