Math 525: Assignment 2 Solutions

1. Note that

$$
\begin{aligned}
\mathbb{P}(A^c \cap B^c) &= \mathbb{P}((A \cup B)^c) \\
&= 1 - \mathbb{P}(A \cup B) \\
&= 1 - (\mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)) \\
&= \mathbb{P}(A^c) - \mathbb{P}(B) + \mathbb{P}(A \cap B) \\
&= \mathbb{P}(A^c) - \mathbb{P}(B) + \mathbb{P}(A)\mathbb{P}(B) \\
&= \mathbb{P}(A^c) - (1 - \mathbb{P}(A))\,\mathbb{P}(B) \\
&= \mathbb{P}(A^c) - \mathbb{P}(A^c)\mathbb{P}(B) \\
&= \mathbb{P}(A^c)\,(1 - \mathbb{P}(B)) \\
&= \mathbb{P}(A^c)\mathbb{P}(B^c).
\end{aligned}
$$

   Similarly, note that

$$
\begin{aligned}
\mathbb{P}(A \cap B^c) &= \mathbb{P}(A \setminus (A \cap B)) \\
&= \mathbb{P}(A) - \mathbb{P}(A \cap B) \\
&= \mathbb{P}(A) - \mathbb{P}(A)\mathbb{P}(B) \\
&= \mathbb{P}(A)\,(1 - \mathbb{P}(B)) \\
&= \mathbb{P}(A)\mathbb{P}(B^c).
\end{aligned}
$$

2. Use the definition of conditional probability:

$$
\begin{aligned}
\mathbb{P}(\text{at least one die is four} \mid \text{sum is seven}) &= \frac{\mathbb{P}(\text{at least one die is four and sum is seven})}{\mathbb{P}(\text{sum is seven})} \\
&= \frac{2\mathbb{P}\left\{(3,4)\right\}}{2\mathbb{P}\left\{(1,6),(2,5),(3,4)\right\}} = \frac{1/36}{3/36} = \frac{1}{3}.
\end{aligned}
$$

3.

   (a) We need only check to make sure $\mathbb{T}$ is a probability measure. First, note that

$$
0 \leq \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \leq \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1
$$

   and hence $0 \leq \mathbb{T}(\cdot) \leq 1$. Moreover,

$$
\mathbb{T}(\emptyset) = \frac{\mathbb{P}(\emptyset \cap B)}{\mathbb{P}(B)} = \frac{0}{\mathbb{P}(B)} = 0
$$

and

$$\mathbb{T}(\Omega) = \frac{\mathbb{P}(\Omega \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1.$$

Lastly, given disjoint sets $A_1, \ldots, A_n \in \mathcal{F}$, note that

$$\begin{aligned}\mathbb{T}(A_1 \cup \cdots \cup A_n) = \frac{\mathbb{P}((A_1 \cup \cdots \cup A_n) \cap B)}{\mathbb{P}(B)} &= \frac{\mathbb{P}((A_1 \cap B) \cup \cdots \cup (A_n \cap B))}{\mathbb{P}(B)} \\ &= \frac{\mathbb{P}(A_1 \cap B) + \cdots + \mathbb{P}(A_n \cap B)}{\mathbb{P}(B)} = \mathbb{T}(A_n).\end{aligned}$$

(b) Conditional probability is only defined if $\mathbb{P}(B) > 0$. Any reasonable definition for the case of $\mathbb{P}(B) = 0$ is uninteresting.

4. Let s_i denote the i-th guest at the table, so that $s_1, \ldots, s_N$ is the entire table. Guests s_i and s_{i+1} are adjacent for any $1 \leq i < N$, but so are guests s_1 and s_N (the table is circular). We use B and K to refer to Barbie and Ken.

   (a) Since there are $N!$ ways for the guests to sit, the answer is

   $$N! - x = \boxed{N(N-3)(N-2)!}$$

   where x is the answer in part (b).

   (b) We break the analysis up into cases:

      i. If $s_1 = B$ and $s_N = K$, Barbie and Ken are adjacent. The other guests can be arranged in $(N-2)!$ ways in this case. We conclude there are $(N-2)!$ arrangements in which Barbie sits at seat s_1 and Ken sits at seat s_N.

      ii. If $s_i = B$ and $s_{i+1} = K$, Barbie and Ken are adjacent. The other guests can once again be arranged in $(N-2)!$ ways. Now, there are $N-1$ ways we can pick i. We conclude there are $(N-1)(N-2)! = (N-1)!$ arrangements in which Barbie sits at seat s_i and Ken sits at seat s_{i+1}.

      iii. Since Barbie and Ken can switch seats and result in a valid arrangement, we have a total of

      $$2((N-2)! + (N-1)!) = \boxed{2N(N-2)!}$$

      possible arrangements in which Barbie and Ken are adjacent.

Note. There is another <u>valid</u> way of interpreting the question. We can consider the arrangement $s_1, \ldots, s_N$ as equivalent to the arrangement $s_N, s_1, s_2, \ldots, s_{N-1}$ obtained by having each guest move to the "next" seat. Alternatively, you can think about this as "rotating" the table while keeping all guests fixed. In this case, the answers to parts (a) and (b) are just divided by N:

$$\boxed{(N-3)(N-2)!} \text{ and } \boxed{2(N-2)!}.$$

5. The base case $n = 1$ is trivial. Suppose the binomial theorem holds for some n. Then,

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)^n (a+b) \\
&= \left(\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \right) (a+b) \\
&= \sum_{k=0}^{n} \binom{n}{k} a^{k+1} b^{n-k} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-(k-1)} \\
&= \sum_{k=1}^{n} \binom{n}{k-1} a^k b^{n-(k-1)} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-(k-1)} \\
&= \sum_{k=1}^{n} \left(\binom{n}{k-1} + \binom{n}{k} \right) a^k b^{n-(k-1)} + \binom{n}{0} a^0 b^{n+1} \\
&= \sum_{k=1}^{n} \binom{n+1}{k} a^k b^{n-(k-1)} + \binom{n}{0} a^0 b^{n+1} \\
&= \sum_{k=1}^{n} \binom{n+1}{k} a^k b^{n+1-k} + \binom{n}{0} a^0 b^{n+1} \\
&= \sum_{k=0}^{n} \binom{n+1}{k} a^k b^{n+1-k}
\end{aligned}$$

where we used the fact that

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!\,(n-k+1)!} + \frac{n!}{k!\,(n-k)!} \\
&= \frac{n!k}{k!\,(n-k+1)!} + \frac{n!\,(n-k+1)}{k!\,(n-k)!} \\
&= \frac{n!\,(k+n-k+1)}{k!\,(n-k+1)!} \\
&= \frac{n!\,(n+1)}{k!\,(n+1-k)!} \\
&= \binom{n+1}{k}.
\end{aligned}$$

6.

(a)

$$\begin{aligned}
x \in f^{-1}(H \cup J) &\iff f(x) \in H \cup J \\
&\iff f(x) \in H \text{ or } f(x) \in J \\
&\iff x \in f^{-1}(H) \text{ or } x \in f^{-1}(J) \\
&\iff x \in f^{-1}(H) \cup f^{-1}(J).
\end{aligned}$$

(b)

$$
\begin{aligned}
x \in f^{-1}(H^c) &\iff f(x) \in H^c \\
&\iff f(x) \notin H \\
&\iff x \notin f^{-1}(H) \\
&\iff x \in \left(f^{-1}(H)\right)^c.
\end{aligned}
$$

(c) By parts (a) and (b),

$$
\begin{aligned}
f^{-1}(H \cap J) &= \left(\left(f^{-1}(H \cap J)\right)^c\right)^c \\
&= \left(f^{-1}(H^c \cup J^c)\right)^c \\
&= \left(f^{-1}(H^c) \cup f^{-1}(J^c)\right)^c \\
&= \left(f^{-1}(H^c)\right)^c \cap \left(f^{-1}(J^c)\right)^c \\
&= f^{-1}(H) \cap f^{-1}(J).
\end{aligned}
$$

7. To show that $X = \sup_n X_n$, it is enough to check that sets of the form $\{X \leq x\}$ are in the underlying σ-algebra, call it $\mathcal{F}$. Note that

$$
\{X \leq x\} = \left\{\sup_n X_n \leq x\right\} = \bigcap_{n \geq 1} \{X_n \leq x\}.
$$

Since each X_n is a random variable, we know that $\{X_n \leq x\} \in \mathcal{F}$. Since $\mathcal{F}$ is closed under countable intersections, the desired result follows.