Midterm 2

Math 525: Probability

Last name, first name: ______________________________

Section number: ______________________________

User ID: ______________________________

Question:	1	2	3	Total
Points:	35	35	30	100
Score:				

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Thursday, March 20, 2018

Question 1 (35 points)

Consider a person walking along a line. Their position at time n is denoted X_n, and they start at position zero (i.e., $X_0 = 0$). At any point in time, the person can either take a step forward ($X_{n+1} = X_n + 1$) or a step backwards ($X_{n+1} = X_n - 1$). Let N be a positive integer and $0 < p < 1$. The person walking follows the rules

$$\begin{aligned}
&\mathbb{P}(X_{n+1} = i+1 \mid X_n = i) = p, && \text{if } -N < i < N \\
&\mathbb{P}(X_{n+1} = i-1 \mid X_n = i) = 1-p, && \text{if } -N < i < N
\end{aligned}$$

and

$$\mathbb{P}(X_{n+1} = \pm(N-1) \mid X_n = \pm N) = 1$$

(a) Write down the transition matrix P for $N = 2$ and $p = 1/3$.

(b) Recalling that $X_0 = 0$, give an expression for $\mathbb{P}(X_{10} = 0)$ using the matrix P. Just write down an expression: do not compute anything!

Question 2 (35 points)

Recall that the characteristic function of a normal random variable $X_n \sim \mathcal{N}(0, n)$ with mean zero and variance n is

$$\phi_n(t) = e^{-nt^2/2}.$$

(a) What is $\phi(t) = \lim_{n\to\infty} \phi_n(t)$?

(b) Can we conclude, using Lévy's continuity theorem, that X_n converges in distribution to a random variable X with characteristic function ϕ? Why or why not?

Lévy's continuity theorem: Let X_n be a random variable with distribution functions F_n and characteristic function ϕ_n. If $\lim_{n\to\infty} \phi_n(t) = \phi(t)$ for some function ϕ which is continuous at the origin, then there exists a distribution function F such that $F_n \Rightarrow F$ and ϕ is the characteristic function of F.

Question 3 (30 points)

We would like to compute

$$\mathbb{E}\left[I_A(X)\right]$$

where A is some (Borel) subset of the real line and X is a random variable. One way to do so involves generating independent samples $X_1, \ldots, X_n$ (each having the same distribution as X) and making the approximation

$$\mathbb{E}\left[I_A(X)\right] \approx \frac{S_n}{n} \qquad \text{where} \qquad S_n = I_A(X_1) + \cdots + I_A(X_n).$$

The central limit theorem tells us

$$\sqrt{n}\left(\frac{S_n}{n} - \mathbb{E}\left[I_A(X)\right]\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2). \tag{*}$$

(a) What is the exact value of σ^2 in the above? Simplify as much as possible.

(b) In your own words, what is the significance of (*)?