

Math 525: Lecture 3

January 11, 2018

1 Random variables

Consider rolling two dice, corresponding to the sample space

$$\Omega = \{(m, n) : 1 \leq m, n \leq 6\}.$$

We can compute various numerical quantities based on the outcome of the rolls. For example, the sum of the two dice

$$X(m, n) = m + n$$

or their product

$$Y(m, n) = mn.$$

As we will see shortly, both X and Y are examples of random variables on a probability space. At least intuitively, a random variable is any function that maps the outcome of an experiment (e.g., (m, n)) to a numerical value (e.g., $m + n$ or mn).

Before we give a rigorous definition of a random variable, let's compute as a motivating example the probability that the two die sum to 5. Letting $X(m, n) = m + n$, this is simply

$$\mathbb{P}(\{\omega \in \Omega : X = 5\}) = \mathbb{P}(\{(1, 4), (2, 3), (3, 2), (4, 1)\}) = 4/36 = 1/9.$$

To give the rigorous definition of a random variable, we review the concept of an inverse map.

Definition 1.1. Let $f: A \rightarrow B$ be a function. Let $H \subset B$. Let

$$f^{-1}(H) = \{a \in A : f(a) \in H\}.$$

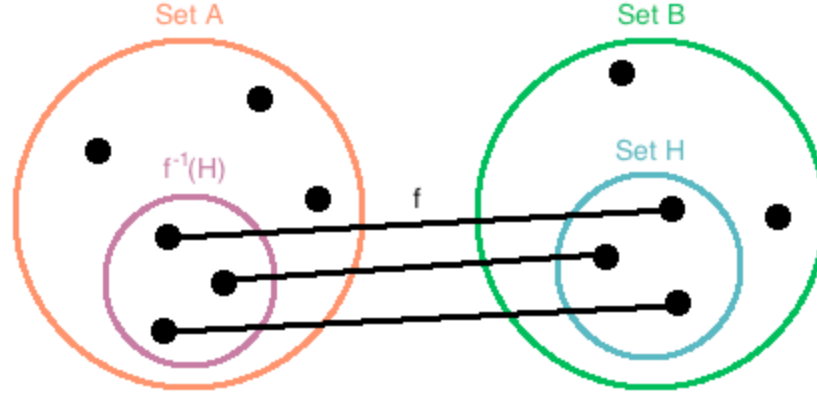
We call f^{-1} the *inverse map* of f . Note that the inverse map does not map points, but rather sets.

We are now ready to give the rigorous definition of a random variable. For the remainder, we will assume an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 1.2. A *random variable* is a function $X: \Omega \rightarrow \mathbb{R}$ which satisfies, for all $B \in \mathcal{B}(\mathbb{R})$,

$$X^{-1}(B) \equiv \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}.$$

The Borel σ -algebra is rather large, so the above definition is not easy to check. The following proposition makes our lives a bit simpler.



The subset H of B has an inverse image $f^{-1}(H)$ which is a subset of A.

Figure 1: Inverse map

Proposition 1.3. *Suppose \mathcal{G} generates the Borel σ -algebra (i.e., $\sigma(\mathcal{G}) = \mathcal{B}(\mathbb{R})$). Then, $X: \Omega \rightarrow \mathbb{R}$ is a random variable if and only if for each generating set $G \in \mathcal{G}$,*

$$X^{-1}(G) \in \mathcal{F}.$$

Proof. We prove only the nontrivial direction. Let

$$\mathcal{M} = \{B \subset \mathbb{R}: X^{-1}(B) \in \mathcal{F}\}.$$

By definition, we know that $\mathcal{G} \subset \mathcal{M}$. Therefore, $\sigma(\mathcal{G}) \subset \sigma(\mathcal{M})$. If \mathcal{M} is a σ -algebra, it follows that

$$\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{G}) \subset \sigma(\mathcal{M}) = \mathcal{M},$$

as desired. To check that \mathcal{M} is a σ -algebra, verify the three properties:

1. $X^{-1}(\emptyset) = \emptyset \in \mathcal{F}$.
2. If $B \in \mathcal{M}$, then $X^{-1}(B^c) = (X^{-1}(B))^c \in \mathcal{F}$.
3. If $B_1, B_2, \dots \in \mathcal{M}$, then $X^{-1}(\cup_{n \geq 1} B_n) = \cup_{n \geq 1} X^{-1}(B_n) \in \mathcal{F}$. □

The above proposition is particularly useful when we take \mathcal{G} to be the set of intervals $\mathcal{G} = \{(-\infty, x]: x \in \mathbb{R}\}$. In this case...

Corollary 1.4. *$X: \Omega \rightarrow \mathbb{R}$ is a random variable if and only if for each $x \in \mathbb{R}$,*

$$\{\omega \in \Omega: X(\omega) \leq x\} \in \mathcal{F}.$$

Proof. If $G = (-\infty, x]$,

$$X^{-1}(G) = X^{-1}((-\infty, x]) = \{\omega \in \Omega: X(\omega) \leq x\}. \quad \square$$

We will often use the above corollary to prove something is a random variable. To simplify notation, we often write

$$\{\omega \in \Omega: X(\omega) \leq x\} = \{X \leq x\}.$$

The above means that for a random variable X , $\mathbb{P}(\{X \leq x\})$ (i.e., the probability that X is at most x) is always well-defined. We define $\{X < x\}$, $\{X = x\}$, $\{X \geq x\}$, and $\{X > x\}$ similarly.

Proposition 1.5. *Let X be a random variable. Then, $\{X < x\}, \{X = x\}, \{X \geq x\}, \{X > x\} \in \mathcal{F}$.*

Proof. Let's just do the case of $\{X < x\}$. We can write

$$\{X < x\} = \bigcup_{\substack{q \in \mathbb{Q} \\ q > 0}} \{X \leq x - q\},$$

in which case we see that $\{X < x\}$ is nothing other than a countable union of sets of the form $\{X \leq a\}$, which we know to be in \mathcal{F} . \square

Proposition 1.6. *Let X and Y be random variables (on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$) and let a be a real number. Then, the following are also random variables:*

1. aX .
2. $X + Y$.
3. XY .
4. Z defined by $Z(\omega) = \begin{cases} Y(\omega)/X(\omega) & \text{if } X(\omega) \neq 0 \\ 0 & \text{if } X(\omega) = 0. \end{cases}$

Proof. Let $x \in \mathbb{R}$ be arbitrary.

1. Note that $\{aX \leq x\} = \{X \leq x/a\}$. Since X is a random variable, $\{X \leq x/a\} \in \mathcal{F}$.
2. It is sufficient to prove that $\{X + Y > x\} \in \mathcal{F}$ since $\{X + Y > x\} = \{X + Y \leq x\}^c$. Note that

$$\{X + Y > x\} = \bigcup_{q \in \mathbb{Q}} \{X > q\} \cap \{Y > x - q\}.$$

In this form, it is clear that $\{X + Y > x\} \in \mathcal{F}$.

3. For a real number a , define

$$a^+ = \max\{a, 0\} \quad \text{and} \quad a^- = \max\{-a, 0\}$$

as its positive and negative parts. Since for any real number a we have $a = a^+ - a^-$, it follows that

$$XY = (X^+ - X^-)(Y^+ - Y^-) = X^+Y^+ + X^+Y^- - X^-Y^+ + X^-Y^-.$$

Therefore, it is sufficient to consider the case in which X and Y are nonnegative. Moreover,

$$\{XY > x\} = \bigcup_{\substack{q \in \mathbb{Q} \\ q > 0}} \{X > q\} \cap \{Y > x/q\}.$$

4. It is sufficient to consider the case of $Y = 1$ since $Y/X = Y(1/X)$. In this case,

$$\begin{aligned} \{Z \leq z\} &= \Omega \cap \{Z \leq z\} \\ &= (\{X \neq 0\} \cup \{X = 0\}) \cap \{Z \leq z\} \\ &= (\{X \neq 0\} \cap \{Z \leq z\}) \cup (\{X = 0\} \cap \{Z \leq z\}) \\ &= (\{X \neq 0\} \cap \{1/X \leq z\}) \cup (\{X = 0\} \cap \{0 \leq z\}). \end{aligned}$$

First, note that $\{0 \leq z\}$ is either equal to \emptyset or Ω , and hence $\{X = 0\} \cap \{0 \leq z\} \in \mathcal{F}$. We leave verifying that $\{X \neq 0\} \cap \{1/X \leq z\} \in \mathcal{F}$ as an exercise. \square

2 Distribution functions

Definition 2.1. The *distribution function* of a random variable X is the function $F: \mathbb{R} \rightarrow [0, 1]$ defined by

$$F(x) = \mathbb{P}(\{X \leq x\}).$$

The distribution function is sometimes called the *cumulative distribution function* (CDF).

Example 2.2. Flip a coin twice. Let X be the number of heads H which show up. Since

$$\begin{aligned} \mathbb{P}(\{TT\}) &= 1/4 \\ \mathbb{P}(\{HT, TH\}) &= 2/4 \\ \mathbb{P}(\{HH\}) &= 1/4, \end{aligned}$$

we have that

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1/4 & \text{if } 0 \leq x < 1 \\ 3/4 & \text{if } 1 \leq x < 2 \\ 1 & \text{if } x \geq 2. \end{cases}$$

Remark 2.3. It now becomes clear why a random variable X requires $\{X \leq x\} \in \mathcal{F}$! It is so that the distribution function is well-defined at any point x .

In terms of the inverse function, note that

$$F(x) = \mathbb{P}(X^{-1}((-\infty, x])).$$

Proposition 2.4. Let X be a random variable and F be its distribution function. Then,

1. F is nondecreasing.

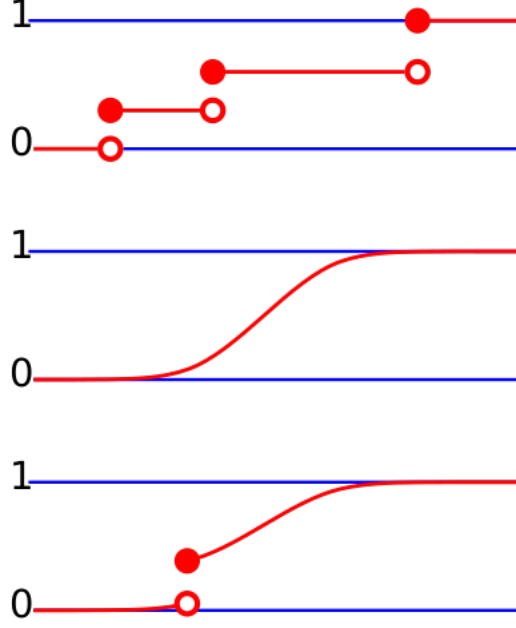


Figure 2: Examples of distribution functions

2. F is right continuous. That is, $F(x) = \lim_{y \downarrow x} F(y)$ for each x .

3. $\lim_{x \rightarrow -\infty} F(x) = 0$ and $\lim_{x \rightarrow \infty} F(x) = 1$.

Proof. Let $A_x = \{X \leq x\}$ so that $F(x) = \mathbb{P}(A_x)$.

1. If $x \leq y$, then $A_x \subset A_y$ and hence $\mathbb{P}(A_x) \leq \mathbb{P}(A_y)$.

2. Let $(y_n)_n$ be a sequence converging to x from above. Then, we have $A_{y_1} \supset A_{y_2} \supset \dots$ and hence $\lim_{n \rightarrow \infty} \mathbb{P}(A_{y_n}) = \mathbb{P}(\bigcap_{n \geq 1} A_{y_n}) = \mathbb{P}(A_x)$.

3. Note that $A_{-1} \supset A_{-2} \supset \dots$ and hence $\lim_{n \rightarrow \infty} \mathbb{P}(A_{-n}) = \mathbb{P}(\bigcap_{n \geq 1} A_{-n}) = \mathbb{P}(\emptyset) = 0$. Similarly, $A_1 \subset A_2 \subset \dots$ and hence $\lim_{n \rightarrow \infty} \mathbb{P}(A_n) = \mathbb{P}(\bigcup_{n \geq 1} A_n) = \mathbb{P}(\Omega) = 1$.

□

Since F is monotone, it follows that F has at most countably many discontinuities. We define

$$F(x-) = \lim_{y \uparrow x} F(y)$$

as the left-hand limit of F .

Proposition 2.5. *Let X be a random variable with distribution function F . Then,*

1. $\mathbb{P}(\{X < x\}) = F(x-)$.

2. $\mathbb{P}(\{X = x\}) = F(x) - F(x-)$.

3. If $a < b$, $\mathbb{P}(\{a < X \leq b\}) = F(b) - F(a)$.

$$4. \mathbb{P}(\{X > x\}) = 1 - F(x).$$

The above implies that if F is continuous, then $\mathbb{P}(\{X = x\}) = 0$ for all x ! This means that there is zero probability of any particular realization of a random variable. Positive probability is assigned only to ranges (e.g., $\{a < X \leq b\}$).

3 Indicator random variables

Definition 3.1. Let $A \in \mathcal{F}$. The *indicator random variable on A* is the function

$$I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A. \end{cases}$$

Note that for an indicator random variable I_A , we have $\mathbb{P}(\{I_A = 1\}) = \mathbb{P}(A)$ and $\mathbb{P}(\{I_A = 0\}) = \mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

Example 3.2. You play a game in which you roll a dice, and if the number you roll is greater than four, you get \$2. Otherwise, you lose \$1. How do we represent your winnings as a random variable?

Let $\Omega = \{1, \dots, 6\}$. Since this is a finite sample space, we can safely take $\mathcal{F} = 2^\Omega$ and define \mathbb{P} by $\mathbb{P}(\{\omega\}) = 1/6$ (each outcome is equally as likely). The random variable corresponding to your winnings is

$$X(\omega) = 2I_{\{\omega > 4\}}(\omega) - I_{\{\omega \leq 4\}}(\omega).$$

We'll often express this more succinctly as

$$X = 2I_{\{\omega > 4\}} - I_{\{\omega \leq 4\}}.$$

Remark 3.3. There are many other common notations for indicator functions. It helps to be aware of these:

$$I_A(\omega) \equiv \mathbf{1}_A(\omega) \equiv \chi_A(\omega) \equiv [A](\omega).$$

4 Borel measurable functions

In closing this section, we introduce the notion of a Borel measurable function, which will give us our final piece of insight as to why we care about Borel σ -algebras.

Definition 4.1. We say $f: \mathbb{R} \rightarrow \mathbb{R}$ is a *Borel measurable function* (a.k.a. Borel function) if for each $B \in \mathcal{B}(\mathbb{R})$,

$$f^{-1}(B) \in \mathcal{B}(\mathbb{R}).$$

That is, the inverse image of a Borel set is also a Borel set.

Proposition 4.2. *If $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous, it is Borel measurable.*

Proof. Let

$$\mathcal{M} = \{B \subset \mathbb{R} : f^{-1}(B) \in \mathcal{B}(\mathbb{R})\}.$$

As usual, we can show that \mathcal{M} is a σ -algebra (check). Let \mathcal{G} be the set of open intervals in \mathbb{R} . Then, since f is continuous, $\mathcal{G} \subset \mathcal{M}$ and as such, $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{G}) \subset \sigma(\mathcal{M}) = \mathcal{M}$ as desired. \square

Exercise 4.3. Let X be a random variable and $f: \mathbb{R} \rightarrow \mathbb{R}$ be a Boreal measurable function. Show that Y defined by $Y(\omega) = f(X(\omega))$ is a random variable.

The above implies, most importantly, that taking the composition of a continuous function f and a random variable X yields a new random variable.