

# Math 525: Lecture 10

February 6, 2018

## 1 Limits of expectations

Remember the moment generating function of a random variable  $X$ ? It looked like

$$M(\theta) = \mathbb{E} [e^{\theta X}].$$

We claimed that  $M(\theta)$  “generates the moments” in the sense that  $M^{(k)}(0) = \mathbb{E}[X^k]$  for  $k \in \mathbb{N}$ . However, we did not justify this claim rigorously, as doing so required us to deal with limits under the expectation. In particular, we wanted to write

$$M(\theta) = \mathbb{E} \left[ \lim_{n \rightarrow \infty} \sum_{n=0}^N \frac{\theta^n}{n!} X^n \right] = \lim_{n \rightarrow \infty} \mathbb{E} \left[ \sum_{n=0}^N \frac{\theta^n}{n!} X^n \right] = \lim_{n \rightarrow \infty} \sum_{n=0}^N \frac{\theta^n}{n!} \mathbb{E} [X^n],$$

but we were unable to justify the second equality above.

Motivated by this (and other applications), we will talk today about limits of expectations. That is, given a sequence of random variables  $(X_n)_n$ , can we conclude that  $\lim_n \mathbb{E}[X_n] = \mathbb{E}[\lim_n X_n]$ ? The answer is, “most of the time, but not always”:

**Example 1.1.** Let  $Y \sim U[0, 1]$  and  $X_n = nI_{[0, 1/n]}(Y)$ . Then,

$$\mathbb{E}[X_n] = n\mathbb{P}(Y \leq 1/n) = 1$$

but  $X_n \rightarrow 0$  a.s.! That is,  $1 = \lim_n \mathbb{E}[X_n] \neq \mathbb{E}[\lim_n X_n] = 0$ .

We start out with a simple limit theorem for expectations. To simplify notation, let  $x \wedge y = \min\{x, y\}$ . For the remainder, it should be understood that all random variables are extended real valued.

**Proposition 1.2.** *Let  $X$  be a random variable such that  $X \geq 0$  a.s. Then,*

$$\lim_{n \rightarrow \infty} \mathbb{E}[X \wedge n] = \lim_{n \rightarrow \infty} \mathbb{E}[X].$$

This is true even when  $X$  is not integrable (i.e.,  $\mathbb{E}[X] = \infty$ ).

*Proof.* Note that

$$X \wedge n = nI_{\{X=\infty\}} + (X \wedge n)I_{\{X<\infty\}}.$$

Therefore, if  $\mathbb{P}\{X = \infty\} > 0$ ,

$$\mathbb{E}[X] \geq \mathbb{E}[X \wedge n] \geq \mathbb{E}[nI_{\{X=\infty\}}] = n\mathbb{P}\{X = \infty\} \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty$$

and the claim is trivially true. Therefore, we may proceed assuming  $X$  is finite everywhere.

Note that

$$\underline{X}_k \wedge n = \underline{X}_k I_{\{\underline{X}_k \leq n\}} + nI_{\{\underline{X}_k > n\}}.$$

Therefore,

$$\begin{aligned} \mathbb{E}[X \wedge n] &\geq \mathbb{E}[\underline{X}_k \wedge n] \\ &\geq \mathbb{E}[\underline{X}_k I_{\{\underline{X}_k \leq n\}}] \\ &= \sum_{j=1}^{2^k n} \frac{j}{2^k} \mathbb{P}\left\{\underline{X}_k = \frac{j}{2^k}\right\}. \end{aligned}$$

If we take limits of both sides of this inequality,

$$\lim_{n \rightarrow \infty} \mathbb{E}[X \wedge n] \geq \sum_{j=1}^{\infty} \frac{j}{2^k} \mathbb{P}\left\{\underline{X}_k = \frac{j}{2^k}\right\} = \mathbb{E}[\underline{X}_k].$$

Taking limits in the above inequality,

$$\lim_{n \rightarrow \infty} \mathbb{E}[X \wedge n] \geq \lim_{k \rightarrow \infty} \mathbb{E}[\underline{X}_k] = \mathbb{E}[X].$$

The reverse inequality is trivial: since  $\mathbb{E}[X] \geq \mathbb{E}[X \wedge n]$  for all  $n$ ,

$$\mathbb{E}[X] \geq \lim_{n \rightarrow \infty} \mathbb{E}[X \wedge n].$$

□

Recall that a nondecreasing sequence of real numbers  $(a_n)_n$  can do one of two things: converge to a finite limit, or diverge to  $\infty$ . The Monotone Convergence Theorem (up next) is the analogue of this claim for random variables (or, more generally, measurable functions).

**Proposition 1.3** (Monotone Convergence Theorem). *Consider a sequence of random variables  $(X_n)_n$  satisfying  $0 \leq X_1 \leq X_2 \leq \dots$  a.s. and  $X_n \rightarrow X$  a.s. Then,*

$$\lim_{n \rightarrow \infty} \mathbb{E}[X_n] = \mathbb{E}[X].$$

As before, this is true even when  $X$  is not integrable (i.e.,  $\mathbb{E}[X] = \infty$ ).

*Proof.* As usual, we can ignore the “a.s.” (though you should convince yourself carefully that this is the case).

Taking expectations of the inequality, we get  $0 \leq \mathbb{E}X_1 \leq \mathbb{E}X_2 \leq \dots$ . Therefore,  $\lim_n \mathbb{E}[X_n] \leq \mathbb{E}[X]$  (note that  $\lim_n \mathbb{E}[X_n]$  may be infinite). Therefore, if we can establish the reverse inequality  $\lim_n \mathbb{E}[X_n] \geq \mathbb{E}[X]$ , we will be done. By Proposition 1.2,  $\mathbb{E}[X_n] = \lim_N \mathbb{E}[X_n \wedge N]$  and  $\mathbb{E}[X] = \lim_N \mathbb{E}[X \wedge N]$  and hence it is sufficient to establish

$$\lim_n \mathbb{E}[X_n \wedge N] \geq \mathbb{E}[X \wedge N] \quad \text{for all } N.$$

Fix  $N$  and let  $Y = X \wedge N$  and  $Y_n = X_n \wedge N$ . Trivially,  $Y$  and  $Y_n$  are integrable. Let  $\epsilon > 0$  and

$$A_{\epsilon,n} = \{Y - Y_n \geq \epsilon\}.$$

Note that  $A_{\epsilon,1} \supset A_{\epsilon,2} \supset \dots$  is a decreasing sequence of sets (since  $Y_1 \leq Y_2 \leq \dots$ ) and hence by continuity of the probability measure,

$$\mathbb{P}(A_{\epsilon,n}) \rightarrow \mathbb{P}(\cap_n A_{\epsilon,n}) \quad \text{as } n \rightarrow \infty.$$

Moreover, since  $Y_n \rightarrow Y$  a.s.,  $\cap_n A_{\epsilon,n} = \emptyset$ . Note that

$$Y - Y_n \leq NI_{A_{\epsilon,n}} + \epsilon I_{A_{\epsilon,n}^c}$$

and hence

$$\mathbb{E}[Y] - \mathbb{E}[Y_n] = \mathbb{E}[Y - Y_n] \leq \mathbb{E}[NI_{A_{\epsilon,n}} + \epsilon I_{A_{\epsilon,n}^c}] \leq N\mathbb{P}(A_{\epsilon,n}) + \epsilon\mathbb{P}(A_{\epsilon,n}^c) \leq N\mathbb{P}(A_{\epsilon,n}) + \epsilon.$$

Taking limits,

$$\mathbb{E}[Y] - \lim_n \mathbb{E}[Y_n] \leq \epsilon.$$

Taking  $\epsilon \rightarrow 0$  gives us the desired inequality.  $\square$

The monotone convergence theorem was for nonnegative increasing sequences. What about decreasing sequences?

**Corollary 1.4.** *Consider a sequence of random variables  $(X_n)_n$  satisfying  $X_1 \geq X_2 \geq \dots \geq 0$  a.s. and  $X_n \rightarrow X$  a.s. If  $X_1$  is integrable, then*

$$\lim_{n \rightarrow \infty} \mathbb{E}[X_n] = \mathbb{E}[X].$$

*Proof.* Let  $Y_n = X_1 - X_n$  and note that  $Y_n$  increases to  $Y = X_1 - X$ . Therefore, by the Monotone Convergence Theorem,

$$\lim_{n \rightarrow \infty} \mathbb{E}[Y_n] = \mathbb{E}[Y].$$

Now,  $\mathbb{E}[Y] = \mathbb{E}[X_1] - \mathbb{E}[X]$  due to integrability. Similarly,  $\mathbb{E}[Y_n] = \mathbb{E}[X_1] - \mathbb{E}[X_n]$ . Plugging these into the above and simplifying, we obtain the desired result.  $\square$

It is not possible to remove the integrability condition from the above:

**Example 1.5.** Let  $Y \sim U[0, 1]$  and  $X_n = \frac{1}{nY}$ . Then,

$$\mathbb{E}[X_n] = \int_0^1 \frac{1}{ny} dy = \frac{1}{n} \int_0^1 \frac{1}{y} dy = \frac{1}{n} \lim_{y \downarrow 0} (\log 1 - \log y) = \infty.$$

However,  $X_n \downarrow 0$  a.s.

**Proposition 1.6** (Fatou's Lemma). *Let  $(X_n)_n$  be a sequence of random variables with  $X_n \geq 0$  a.s. Then,*

$$\liminf_{n \rightarrow \infty} \mathbb{E}[X_n] \geq \mathbb{E} \left[ \liminf_{n \rightarrow \infty} X_n \right].$$

*Proof.* Trivially,

$$\mathbb{E}[X_n] \geq \mathbb{E} \left[ \inf_{j \geq n} X_j \right].$$

Letting  $Y_n = \inf_{j \geq n} X_j$  and applying limit inferiors to both sides of the above inequality,

$$\liminf_n \mathbb{E}[X_n] \geq \liminf_n \mathbb{E}[Y_n].$$

Note, in particular, that  $Y_n$  is a nondecreasing sequence. Therefore, by the Monotone Convergence Theorem,

$$\lim_n \mathbb{E}[Y_n] = \mathbb{E} \left[ \lim_n Y_n \right] = \mathbb{E} \left[ \liminf_n X_j \right] = \mathbb{E} \left[ \liminf_n X_n \right].$$

□

**Proposition 1.7.** *Let  $(X_n)_n$  be a sequence of random variables dominated by some integrable random variable  $Y$  (i.e.,  $\mathbb{E}|Y| < \infty$  and  $|X_n| \leq Y$ ) such that  $X_n \rightarrow X$  a.s. Then,*

$$\mathbb{E}[X_n] \rightarrow \mathbb{E}[X].$$

*Proof.* First, we handle the a.s. case. Indeed, suppose  $X_n \rightarrow X$  a.s. Then,  $Y - X_n \geq 0$  a.s. and  $Y - X_n \rightarrow Y - X$  a.s. By Fatou's lemma,

$$\begin{aligned} \mathbb{E}[Y] - \limsup_n \mathbb{E}[X_n] &= \liminf_n \mathbb{E}[Y - X_n] \geq \mathbb{E} \left[ \liminf_n (Y - X_n) \right] = \mathbb{E}[Y] - \mathbb{E} \left[ \limsup_n X_n \right] \\ &= \mathbb{E}[Y] - \mathbb{E} \left[ \lim_n X_n \right]. \end{aligned}$$

Therefore,

$$\limsup_n \mathbb{E}[X_n] \leq \mathbb{E} \left[ \lim_n X_n \right].$$

An identical argument with  $Y + X_n \geq 0$  yields

$$\liminf_n \mathbb{E}[X_n] \geq \mathbb{E} \left[ \lim_n X_n \right].$$

Combining the two inequalities above, the desired result follows. □

**Corollary 1.8.** *Let  $(X_n)_n$  be a sequence of random variables dominated by a real number (i.e.,  $|X_n| \leq M$ ) such that  $X_n \rightarrow X$  a.s. Then,*

$$\mathbb{E}[X_n] \rightarrow \mathbb{E}[X].$$

*Proof.* Take  $Y = M$  in the Dominated Convergence Theorem. □

**Corollary 1.9.** *Let  $(X_n)_n$  be a sequence of random variables satisfying the requirements of the Dominated Convergence Theorem. Then,  $X_n \xrightarrow{\mathbb{L}^1} X$ .*

*Proof.* Note that  $|X_n - X| \leq 2Y$  and  $|X_n - X| \rightarrow 0$ . Therefore,  $\mathbb{E}|X_n - X| \rightarrow 0$  by the Dominated Convergence Theorem, and hence the sequence converges in  $\mathbb{L}^1$ . □

Conversely, if we have a sequence  $(X_n)_n$  converging to some  $X$  in  $\mathbb{L}^1$ ,

$$|\mathbb{E}X_n - \mathbb{E}X| = \mathbb{E}[|X_n - X|] \rightarrow 0$$

and hence it is trivially the case that  $\mathbb{E}X_n \rightarrow \mathbb{E}X$ .