

Math 525: Assignment 9

1. Let $X, Y \sim U[0, 1]$ be independent. Find the joint density of $\min(X, Y)$ and $\max(X, Y)$.
2. Let X and Y be random variables admitting the joint density

$$f_{XY}(x, y) = \begin{cases} \frac{1}{\pi} \exp\left(-\frac{(x^2 + y^2)}{2}\right) & \text{if } xy \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Show that $X, Y \sim \mathcal{N}(0, 1)$ but that X and Y are not independent. **Hint:** compute the marginal distribution.

3. Let U and V be independent $\mathcal{N}(0, 1)$ random variables. Let $X = aU + bV$ and $Y = cU + dV$ where $a^2 + b^2 = 1$, $c^2 + d^2 = 1$, and $ac + bd = \rho$ with $|\rho| < 1$.
 - (a) Show that $X \sim \mathcal{N}(0, 1)$. **Hint:** what is the characteristic function of X ?
 - (b) Show that $\mathbb{E}[XY] = \rho$.
 - (c) The *joint characteristic function* of X and Y is a function $\varphi : \mathbb{R}^2 \rightarrow \mathbb{C}$ given by

$$\varphi_{XY}(t, s) = \mathbb{E}[\exp(itX + isY)]$$

(this can be generalized to the case of n random variables by $\varphi_{X_1 \dots X_n}(t_1, \dots, t_n) = \mathbb{E}[\exp(it_1 X_1 + \dots + it_n X_n)]$). Show that

$$\varphi_{XY}(t, s) = \exp\left(-\frac{t^2 + 2\rho ts + s^2}{2}\right).$$

- (d) Show that the joint density of X and Y is given by

$$f_{XY}(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right).$$

Hint: use the fact that if X and Y admit a joint density, it can be obtained as the *inverse Fourier transform* of their joint characteristic function (see page 152 of Walsh for details):

$$f_{XY}(x, y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_{X,Y}(t, s) \exp(-i(tx + sy)) dt ds.$$

- (e) Part (b) implies that if $\rho \neq 0$, then X and Y are not independent. Use your findings in part (d) to establish the converse. **Hint:** recall that $f_{XY} = f_X f_Y$ is a sufficient condition for independence.