Math 525: Assignment 7

1. Let $X_n, Y_n \sim B(n, \lambda/n)$ be independent Binomial random variables. Let $Z_n = X_n - Y_n$. Show that Z_n converges in distribution to some random variable Z with characteristic function
$$\phi(t) = e^{2\lambda(\cosh t - 1)}.$$
Actually, this is called a Skellam random variable (look it up on Wikipedia).

2. Ten numbers $X_1, \ldots, X_{10}$ are rounded to the nearest integer $[X_1], \ldots, [X_{10}]$ and then summed $[X_1] + \ldots + [X_{10}]$. Assume the errors from rounding $Y_j = X_j - [X_j]$ are independent and uniformly distributed in $[-1/2, 1/2]$. Use the CLT to determine the approximate probability that the error
$$|X_1 + \cdots + X_{10} - ([X_1] + \cdots + [X_{10}])|$$
is no greater than one.

3. Let $(X_n)_{n\geq 0}$ be a Markov chain. Show that $((X_n, X_{n+1}))_{n\geq 0}$ is also a Markov chain.

4.
   (a) Show that if P is a transition matrix, then P^2 is also a transition matrix. Use this to conclude, by induction, that P^n is a transition matrix for any n.

   (b) A matrix $P = (P_{ij})$ is *bistochastic* if it satisfies (1) $P_{ij} \geq 0$, (2) $\sum_j P_{ij} = 1$, and (3) $\sum_i P_{ij} = 1$ for all j. Show that if P is bistochastic, then P^2 is also bistochastic. Use this to conclude, by induction, that P^n is bistochastic.

5.
   (a) Recall the matrix P for the gambler's ruin with total wealth $N = 4$ and probability of victory $p = 1/2$:
$$P = \begin{pmatrix} 1 & 0 & & & \\ 1/2 & 0 & 1/2 & & \\ & 1/2 & 0 & 1/2 & \\ & & 1/2 & 0 & 1/2 \\ & & & 0 & 1 \end{pmatrix}.$$
   Compute the eigenvalues of P. What is the multiplicity of the eigenvalue 1?

(b) Now, consider a similar, but slightly different transition matrix:

$$P' = \begin{pmatrix} 0 & 1 & & & \\ 1/2 & 0 & 1/2 & & \\ & 1/2 & 0 & 1/2 & \\ & & 1/2 & 0 & 1/2 \\ & & & 0 & 1 \end{pmatrix}.$$

We can interpret this as follows: when the gambler loses, the opponent is kind and gives the gambler a dollar so that they can keep playing. Compute the eigenvalues of P'. What is the multiplicity of the eigenvalue 1?

(c) What do you think the multiplicity of the eigenvalue 1 tells you about the Markov chain?