Math 525: Assignment 4

1. (Moment generating functions) Let X be a random variable with distribution function
$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$$
(we call X a *standard normal random variable*; you may have seen it before). Compute the third raw moment of X using the moment generating function.
Hint: If X is an absolutely continuous distribution function F with density f (i.e., $F(x) = \int_{-\infty}^{x} f(y)dy$) and if $g \geq 0$ is a Borel measurable function, then
$$\mathbb{E}\left[g(X)\right] = \int_{-\infty}^{\infty} g(u)f(u)du.$$

2. (Expectations) Let X be a nonnegative integrable random variable.
   (a) Show that $X = 0$ a.e. implies $\mathbb{E}X = 0$.
   (b) Show that $\mathbb{E}X = 0$ implies $X = 0$ a.e. **Hint**: consider sets of the form $A_n = \{X \geq 1/n\}$ where n is a positive integer.
   (c) Let E be a set of probability zero (i.e., $\mathbb{P}(E) = 0$). Use part (a) to show that $\mathbb{E}[XI_E] = 0$ where I_E is the indicator random variable on E.

3. (Expectations) Let X be a nonnegative integrable random variable with distribution function F.
   (a) Show that, for any real number $x \geq 0$,
   $$x = \int_{0}^{\infty} I_{[0,x)}(y)dy$$
   where $I_{[0,x)}(y) = 1$ if $0 \leq y < x$ and $I_{[0,x)}(y) = 0$ otherwise. Use this to derive
   $$\mathbb{E}X = \mathbb{E}\left[\int_{0}^{\infty} I_{[0,X)}(y)dy\right].$$
   (b) (Optional) Show, using the results of part (a), that
   $$\mathbb{E}X = \int_{0}^{\infty} \mathbb{E}\left[I_{[0,X)}(y)\right] dy.$$
   (c) Show that $\mathbb{E}[I_{[0,X)}(y)] = \mathbb{P}(X > y)$.
   (d) Combine your findings in parts (b) and (c) to conclude
   $$\mathbb{E}X = \int_{0}^{\infty} (1 - F(y))\, dy.$$