Math 525: Assignment 3

1. (Borel measurable) Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous.

   (a) Let
   $$\mathcal{G} = \{(a,b)\colon -\infty < a < b < \infty\}$$
   be the set of all open intervals. Show $\sigma(\mathcal{G}) = \mathcal{B}(\mathbb{R})$.[1]

   (b) Define
   $$\mathcal{M} = \left\{B \subset \mathbb{R}\colon f^{-1}(B) \in \mathcal{B}(\mathbb{R})\right\}.$$
   Show that $\mathcal{M}$ is a σ-algebra.[2] This establishes $\sigma(\mathcal{M}) = \mathcal{M}$.

   (c) (Optional) Show that for any $G \in \mathcal{G}$, $f^{-1}(G)$ is a countable union of open intervals. This establishes $\mathcal{G} \subset \mathcal{M}$.

   (d) Use (a), (b), and (c) to conclude that $\mathcal{B}(\mathbb{R}) \subset \mathcal{M}$ and hence f is Borel measurable.

2. (Distribution function) Let X be a discrete random variable with distribution function F. Show that $\sum_n F(x_n) - F(x_n-) = 1$.

3. (Uniform random variable) Define the function $f\colon \mathbb{R} \to \mathbb{R}$ by
   $$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$
   and the function $F\colon \mathbb{R} \to [0,1]$ by $F(x) = \int_{-\infty}^{x} f(y)dy$. Let $Y \sim U[0,1]$. Note that $F^{-1}(\{0\})$ is a set contaning more than one element. Moreover, $F^{-1}(\{1\})$ is empty. As such, F is not technically a bijection, and hence the expression $F^{-1}(Y)$ is not well-defined. However, note that $\{Y = 0\}$ and $\{Y = 1\}$ occur with zero probability. Therefore, assuming Y does not take the values 0 or 1, we can unambiguously define $X = F^{-1}(Y)$.

   (a) Simplify the expression $X = F^{-1}(Y)$ as much as possible.

   (b) What is the distribution function of X?

[1] **Hint**: use the fact that $(x,\infty) = (x,x+2) \cup (x+1,x+3) \cup \cdots$ and $(-\infty,x] = \mathbb{R} \setminus (x,\infty)$

[2] **Hint**: use the properties of f^{-1} from the previous assignment

4. (Expectation) Let $X_1, X_2, \ldots$ be nonnegative integer-valued random variables. Suppose they are independent and have the same distribution function F (in this case, we say $X_1, X_2, \ldots$ are *independent and identically distributed*, or i.i.d. for short). Show that[3]

$$\mathbb{E}\min\{X_1, \ldots, X_m\} = \sum_{n=1}^{\infty} \mathbb{P}\{X_1 \geq n\}^m.$$

5. (Integrability) Let X be a discrete random variable. Suppose X^2 is integrable. Show that X is integrable.[4]

[3]**Hint**: use the fact that $\mathbb{E}Y = \sum_{n\geq 1} \mathbb{P}(\{Y \geq n\})$ for a nonnegative integer-valued random variable Y

[4]**Hint**: use the inequality $|a| \leq \max\{1, |a|^2\}$