Math 525: Assignment 2

1. (Independence) Let A and B be independent events. Show that $A^c = \Omega \setminus A$ and $B^c = \Omega \setminus B$ are independent. Also show that A and B^c are independent (and so too, by symmetry, are A^c and B).

2. (Conditional probability) Roll two (fair) dice. What is the probability that at least one of the two dice is four given that their sum is seven?

3. (Conditional probability) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $B \in \mathcal{F}$ be an event with $\mathbb{P}(B) > 0$. Define the function $\mathbb{T} \colon \mathcal{F} \to [0, 1]$ by $\mathbb{T}(A) = \mathbb{P}(A \mid B)$.

    (a) Show that $(\Omega, \mathcal{F}, \mathbb{T})$ is a probability space. **Hint**: since we already know that $\mathcal{F}$ is a σ-algebra on Ω, we need only check that $\mathbb{T}$ is a probability measure.

    (b) Why did we require $\mathbb{P}(B) > 0$?

4. (Counting) Consider a circular table with $N \geq 4$ seats. Barbie and Ken are two of N dinner guests to be seated at this table. In how many ways can the dinner guests be seated such that...

    (a) Barbie and Ken are not adjacent.

    (b) Barbie and Ken are adjacent.

5. (Counting) Prove the binomial theorem. That is, prove that for a positive integer n and real numbers a and b,

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

    Hint: an easy way to do this is with induction.

6. (Inverse images) Let $f \colon A \to B$. Show that for any $H, J \subset B$,

    (a) $f^{-1}(H \cup J) = f^{-1}(H) \cup f^{-1}(J)$.

    (b) $f^{-1}(H^c) = (f^{-1}(H))^c$.

    (c) $f^{-1}(H \cap J) = f^{-1}(H) \cap f^{-1}(J)$. **Hint**: use parts (a) and (b).

7. (Random variables) Let $(X_n)_{n \geq 1}$ be a sequence of random variables. Prove that $\sup_n X_n$ is also a random variable.