

# Math 525: Assignment 1

1. Sometimes it is useful to start with a small set of events  $\mathcal{G}$  that is not necessarily a  $\sigma$ -algebra and “generate” a  $\sigma$ -algebra from it. The  $\sigma$ -algebra generated from  $\mathcal{G}$  is

$$\sigma(\mathcal{G}) = \bigcap_{\substack{\mathcal{F} \text{ is a } \sigma\text{-algebra on } \Omega \\ \mathcal{G} \subset \mathcal{F}}} \mathcal{F}.$$

That is,  $\sigma(\mathcal{G})$  is the intersection of all  $\sigma$ -algebras containing  $\mathcal{G}$ . Show that...

- (a)  $\sigma(\mathcal{G})$  is a  $\sigma$ -algebra.
  - (b)  $\sigma(\mathcal{G}) \subset \sigma(\mathcal{G}')$  whenever  $\mathcal{G} \subset \mathcal{G}'$ .
  - (c) If  $\mathcal{F}$  is a  $\sigma$ -algebra, then  $\sigma(\mathcal{F}) = \mathcal{F}$ .
  - (d) If  $\mathcal{F}$  is a  $\sigma$ -algebra and  $\mathcal{G} \subset \mathcal{F}$ , then  $\sigma(\mathcal{G}) \subset \mathcal{F}$ .
2. Show that the following two are algebras but not  $\sigma$ -algebras:
- (a) All finite subsets of  $\mathbb{R}$  together with their complements.
  - (b) All finite unions of intervals in  $\mathbb{R}$  of the form  $(a, b]$ ,  $(-\infty, a]$ , and  $(b, \infty)$ .
3. Prove the principle of inclusion-exclusion. That is, show that if  $A_1, \dots, A_n \in \mathcal{F}$ , then

$$\begin{aligned} & \mathbb{P}(A_1 \cup \dots \cup A_n) \\ &= \sum_i \mathbb{P}(A_i) - \sum_{i < j} \mathbb{P}(A_i \cap A_j) + \sum_{i < j < k} \mathbb{P}(A_i \cap A_j \cap A_k) - \dots + (-1)^{n+1} \mathbb{P}(A_1 \cap \dots \cap A_n). \end{aligned}$$

Note that when we write  $\sum_{i < j}$ , we mean  $\sum_{i, j \in \{(i, j) : i < j\}}$  (and similarly for the sums involving more indices).