Math 525: Assignment 1

1. Sometimes it is useful to start with a small set of events $\mathcal{G}$ that is not necessarily a σ-algebra and "generate" a σ-algebra from it. The σ-algebra generated from $\mathcal{G}$ is

$$\sigma(\mathcal{G}) = \bigcap_{\substack{\mathcal{F} \text{ is a } \sigma\text{-algebra on } \Omega \\ \mathcal{G} \subset \mathcal{F}}} \mathcal{F}.$$

   That is, $\sigma(\mathcal{G})$ is the intersection of all σ-algebras containing $\mathcal{G}$. Show that...

   (a) $\sigma(\mathcal{G})$ is a σ-algebra.

   (b) $\sigma(\mathcal{G}) \subset \sigma(\mathcal{G}')$ whenever $\mathcal{G} \subset \mathcal{G}'$.

   (c) If $\mathcal{F}$ is a σ-algebra, then $\sigma(\mathcal{F}) = \mathcal{F}$.

   (d) If $\mathcal{F}$ is a σ-algebra and $\mathcal{G} \subset \mathcal{F}$, then $\sigma(\mathcal{G}) \subset \mathcal{F}$.

2. Show that the following two are algebras but not σ-algebras:

   (a) All finite subsets of $\mathbb{R}$ together with their complements.

   (b) All finite unions of intervals in $\mathbb{R}$ of the form $(a, b]$, $(-\infty, a]$, and (b, ∞).

3. Prove the principle of inclusion-exclusion. That is, show that if $A_1, \ldots, A_n \in \mathcal{F}$, then

$$\begin{aligned}&\mathbb{P}(A_1 \cup \cdots \cup A_n) \\ &= \sum_i \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} \mathbb{P}(A_1 \cap \cdots \cap A_n).\end{aligned}$$

   Note that when we write $\sum_{i<j}$, we mean $\sum_{i,j \in \{(i,j) \colon i<j\}}$ (and similarly for the sums involving more indices).